Inspiring Greatness
INDEPENDENCE SCHOOL DISTRICT

Math Virtual Learning

Calculus AB

Exponential Growth/Decay and Differential Equations

April 17, 2020

Calculus AB

Lesson: April 17, 2020

Objective/Learning Target:

Students will identify exponential growth and decay models from a differential equation

Warm-Up:

Watch Videos: Exponential Models and Differential Equations
More Exponential Models

Read Article: Differential Equations: Growth/Decay

Notes:

EXPONENTIAL GROWTH AND DECAY MODELS

If y is a differentiable function of t such that $y > 0$ and $y' = ky$ for some constant k, then

$$y = Ce^{kt}$$

C is the **initial value** of y, and k is the **proportionality constant. Exponential growth** occurs when $k > 0$, and **exponential decay** occurs when $k < 0$.

Examples:

A herd of llamas has 1000 llamas in it, and the population is growing exponentially. At time $t = 4$ it has 2000 llamas. Write a formula for the number of llamas at *arbitrary* time t.

Solution: Here there is no direct mention of differential equations, but use of the buzz-phrase *'growing exponentially'* must be taken as indicator that we are talking about the situation

$$f(t) = ce^{kt}$$

where here $f(t)$ is the number of llamas at time t and c, k are constants to be determined from the information given in the problem. And the use of language should probably be taken to mean that at time $t = 0$ there are 1000 llamas, and at time $t = 4$ there are 2000. Then, either repeating the method above or plugging into the formula derived by the method, we find

$$c = \text{ value of } f \text{ at } t = 0 \; = 1000$$

$$k = \frac{\ln f(t_1) - \ln f(t_2)}{t_1 - t_2} = \frac{\ln 1000 - \ln 2000}{0 - 4}$$

$$= \ln \frac{1000}{2000} - 4 = \frac{\ln \frac{1}{2}}{-4} = (\ln 2)/4$$

Therefore,

$$f(t) = 1000\, e^{\frac{\ln 2}{4} t} = 1000 \cdot 2^{t/4}$$

This is the desired formula for the number of llamas at arbitrary time t.

Examples:

A colony of bacteria is growing exponentially. At time $t = 0$ it has 10 bacteria in it, and at time $t = 4$ it has 2000. At what time will it have 100, 000 bacteria?

Solution: Even though it is not explicitly demanded, we need to find the general formula for the number $f(t)$ of bacteria at time t, set this expression equal to 100, 000, and solve for t. Again, we can take a *little* shortcut here since we know that $c = f(0)$ and we are given that $f(0) = 10$. (This is easier than using the bulkier more general formula for finding c). And use the formula for k:

$$k = \frac{\ln f(t_1) - \ln f(t_2)}{t_1 - t_2} = \frac{\ln 10 - \ln 2,000}{0 - 4} = \frac{\ln \frac{10}{2,000}}{-4} = \frac{\ln 200}{4}$$

Therefore, we have

$$f(t) = 10 \cdot e^{\frac{\ln 200}{4} t} = 10 \cdot 200^{t/4}$$

as the general formula. Now we try to solve

$$100,000 = 10 \cdot e^{\frac{\ln 200}{4} t}$$

for t: divide both sides by the 10 and take logarithms, to get

$$\ln 10,000 = \frac{\ln 200}{4} t$$

Thus,

$$t = 4 \frac{\ln 10,000}{\ln 200} \approx 6.953407835.$$

Practice:

1) Ten grams of the plutonium isotope ^{239}Pu were released in a nuclear accident. How long will it take for the 10 grams to decay to 1 gram?

2) An experimental population of fruit flies increases according to the law of exponential growth. There were 100 flies after the second day of the experiment and 300 flies after the fourth day. Approximately how many flies were in the original population?

Answer Key:

Once you have completed the problems, check your answers here.

1) So, the model is

$$y = 10e^{-0.000028761t}. \quad \text{Half-life model}$$

To find the time it would take for 10 grams to decay to 1 gram, you can solve for t in the equation

$$1 = 10e^{-0.000028761t}.$$

The solution is approximately 80,059 years.

2) So, the exponential growth model is

$$y = Ce^{0.5493t}.$$

To solve for C, reapply the condition $y = 100$ when $t = 2$ and obtain

$$100 = Ce^{0.5493(2)}$$

$$C = 100e^{-1.0986}$$

$$C \approx 33.$$

So, the original population (when $t = 0$) consisted of approximately $y = C = 33$ flies,

Additional Practice:

[Interactive Practice]

[Extra Practice with Answers]